REGULARITY OF WEAK SOLUTIONS OF A COMPLEX MONGE-AMPÈRE EQUATION

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Abstract. We prove the smoothness of weak solutions to an elliptic complex Monge-Ampère equation, using the smoothing property of the corresponding parabolic flow.

1. Introduction

Let (M,ω) be a compact Kähler manifold. Our main result is the following.

Theorem 1.1. *Suppose that $\varphi \in PSH(M,\omega)\cap L^\infty(M)$ is a solution of the equation*

$$(\omega+\sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{-F(\varphi,z)}\omega^n$$

in the sense of pluripotential theory [2], *where $F : \mathbf{R}\times M \to \mathbf{R}$ is smooth. Then φ is smooth.*

In particular if M is Fano, $\omega \in c_1(M)$ and h_ω satisfies $\sqrt{-1}\partial\overline{\partial}h_\omega = \mathrm{Ric}(\omega)-\omega$ then we can set $F(\varphi,z)=\varphi-h_\omega$. The result then implies that Kähler-Einstein currents with bounded potentials are in fact smooth. Such weak Kähler-Einstein metrics were studied by Berman-Boucksom-Guedj-Zeriahi in [3], as part of their variational approach to complex Monge-Ampère equations.

It follows from Kołodziej [13] (see also [9]) that the solution φ in Theorem 1.1 is automatically C^α for some $\alpha>0$, but it does not seem possible to use this directly to get further regularity. The difficulty is that in the equation

$$(\omega+\sqrt{-1}\partial\overline{\partial}\varphi)^n = e^f\omega^n,$$

the C^1 estimate for φ (due to Błocki [4] and Hanani [10]) depends on a C^1 bound for f, and in turn the Laplacian estimate for φ (due to Yau [19] and Aubin [1]) depends on the Laplacian of f.

To get around this difficulty we look at the corresponding parabolic flow

$$\frac{\partial\varphi}{\partial t} = \log\frac{(\omega+\sqrt{-1}\partial\overline{\partial}\varphi)^n}{\omega^n} + F(\varphi,z).$$

Following the construction of Song-Tian [18] for the Kähler-Ricci flow, we show that to find a solution for a short time, it is enough to have a C^0 initial condition φ_0 for which $(\omega+\sqrt{-1}\partial\overline{\partial}\varphi_0)^n$ is bounded (see also [6, 7, 8] for earlier results, as well as [17] for a weaker statement in the Riemannian case). The solution of the flow will be smooth at any positive time. Then we need to argue that if the initial condition φ_0 is a weak solution of the elliptic problem then the flow is stationary, so in fact φ_0 is smooth.

In Section 2 we show that the flow (with smooth initial data) exists for a short time, which only depends on a bound for $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$. In Section 3 we

use this to construct a solution to the flow with rough initial data, and we prove Theorem 1.1.

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2. Existence for the parabolic equation

In this section we consider the parabolic equation

$$\frac{\partial\varphi}{\partial t} = \log\frac{(\omega+\sqrt{-1}\partial\overline{\partial}\varphi)^n}{\omega^n} + F(\varphi, z), \tag{1}$$

where $F : \mathbf{R}\times M \to \mathbf{R}$ is smooth and we have the smooth initial condition $\varphi|_{t=0} = \varphi_0$. We write $\dot{\varphi_0}$ for $\frac{\partial}{\partial t}\varphi$ at $t = 0$.

The main result of this section is the following

Proposition 2.1. *There exist $T > 0$ depending only on $\sup|\varphi_0|, \sup|\dot{\varphi_0}|$ (and ω and F), such that there is a smooth solution $\varphi(t, z) : [0, T]\times M \to \mathbf{R}$ to Equation (1). Moreover we also have smooth functions $C_k : (0, T] \to \mathbf{R}$ depending only on $\sup|\varphi_0|, \sup|\dot{\varphi_0}|$ such that*

$$\|\varphi(t)\|_{C^k(M)} < C_k(t) \tag{2}$$

as long as $t \leqslant T$. Note that $C_k(t)\to\infty$ as $t\to 0$.

The proof of the C^1 estimate is based on the arguments in Błocki [4] (see also [10, 16]), whereas the C^2 estimate is based on the Aubin-Yau second order estimate [1, 19] (see also [18] for the parabolic version we need here). The C^3 and higher order estimates follow the standard arguments in [19, 5, 14], although there are a few new terms to control.

The existence of a smooth solution for $t \in [0, T')$ for some $T' > 0$ that depends on the $C^{2,\alpha}$ norm of φ_0 is standard. The aim is to obtain the estimates (2), which allow us to extend the solution up to a time T, which only depends on the initial condition in a weaker way. We will write $\varphi(t)$ for the short time solution.

Lemma 2.1. *There exists $T, C > 0$ depending only on $\sup|\varphi_0|$ and $\sup|\dot{\varphi_0}|$ such that*

$$|\varphi(t)|, |\dot{\varphi}(t)| < C, \tag{3}$$

as long as the solution exists and $t \leqslant T$. In particular

$$\left|\log\frac{(\omega+\sqrt{-1}\partial\overline{\partial}\varphi)^n}{\omega^n}\right| < C \tag{4}$$

for $t \leqslant T$.

Proof. For all s let us define

$$\overline{F}(s) = \sup_{z\in M} F(s,z),$$
$$\underline{F}(s) = \inf_{z\in M} F(s,z),$$

which are continuous functions. If $M_t = \sup \varphi(t,\cdot)$ and $m_t = \inf \varphi(t,\cdot)$ then we obtain

$$\frac{dM_t}{dt} \leqslant \overline{F}(M_t),$$
$$\frac{dm_t}{dt} \geqslant \underline{F}(m_t).$$

Comparing with the corresponding ODEs, we find that there exist $T, C > 0$ depending only on m_0, M_0 such that as long as our solution exists, and $t \leqslant T$, we have $\sup |\varphi(t)| < C$.

Differentiating the equation we obtain

$$\frac{\partial \dot\varphi}{\partial t} = \Delta_\varphi \dot\varphi + F'(\varphi, z)\dot\varphi, \tag{5}$$

where F' is the derivative of F with respect to the φ variable. Since $F'(\varphi,z)$ is bounded as long as φ is bounded, from the maximum principle we get

$$\sup |\dot\varphi(t)| < \sup |\dot\varphi(0)| e^{\kappa t}, \tag{6}$$

where κ depends on F and $\sup|\varphi(0)|$. Hence for our choice of T, we get

$$\sup |\dot\varphi(t)| < C,$$

for $t \leqslant T$, where C depends on $\sup|\varphi_0|$ and $\sup|\dot\varphi_0|$. □

In the lemmas below T will be the same as in the previous lemma.

Lemma 2.2. *There exists $C > 0$ depending on $\sup|\varphi_0|$ and $\sup|\dot\varphi_0|$ such that*

$$|\nabla\varphi(t)|^2_\omega < e^{C/t}, \tag{7}$$

as long as the solution exists and $t \leqslant T$ for the T in Lemma 2.1.

Proof. We modify Błocki's estimate [4] for the complex Monge-Ampère equation (cfr. [10]). Define

$$K = t \log |\nabla\varphi|^2_\omega - \gamma(\varphi),$$

where γ will be chosen later. Suppose that $\sup_{(0,t]\times M} K = K(t,z)$ is achieved. Pick normal coordinates for ω at z, such that $\varphi_{i\bar{j}}$ is diagonal at this point (here and henceforth, indices will denote covariant derivatives with respect to the metric ω). Let us write $\beta = |\nabla\varphi|^2_\omega$ and Δ_φ for the Laplacian of the metric $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$. There exists $B > 0$ such that

$$0 \leqslant \left(\frac{\partial}{\partial t} - \Delta_\varphi\right) K \leqslant -\frac{t}{\beta}\sum_{i,p} \frac{|\varphi_{ip}|^2 + |\varphi_{i\bar{p}}|^2}{1+\varphi_{p\bar{p}}} + (t^{-1}(\gamma')^2 + \gamma'')\sum_p \frac{|\varphi_p|^2}{1+\varphi_{p\bar{p}}}$$
$$-(\gamma' - Bt)\sum_p \frac{1}{1+\varphi_{p\bar{p}}} + \log\beta + \frac{Ct}{\beta} - \gamma'\dot\varphi + n\gamma' + Ct.$$

The constant C depends on bounds for F and F', and also we used that $\nabla K = 0$ at (t,z).

Now we apply Błocki's trick to get rid of the term containing $(\gamma')^2$. At (t, z) we have

$$t\beta_p = \gamma'\beta\varphi_p,$$

where

$$\beta_p = \varphi_p\varphi_{p\bar{p}} + \sum_j \varphi_{jp}\varphi_{\bar{j}},$$

remembering that $\varphi_{j\bar{p}}$ is diagonal. It follows that

$$\sum_j \varphi_{jp}\varphi_{\bar{j}} = (t^{-1}\gamma'\beta - \varphi_{p\bar{p}})\varphi_p,$$

and so

$$\begin{aligned}
\frac{t}{\beta}\sum_{j,p}\frac{|\varphi_{jp}|^2}{1+\varphi_{p\bar{p}}} &\geqslant \frac{t}{\beta^2}\sum_p \frac{\left|\sum_j \varphi_{jp}\varphi_{\bar{j}}\right|^2}{1+\varphi_{p\bar{p}}} \\
&= \frac{t}{\beta^2}\sum_p \frac{|t^{-1}\gamma'\beta - \varphi_{p\bar{p}}|^2|\varphi_p|^2}{1+\varphi_{p\bar{p}}} \\
&\geqslant t^{-1}(\gamma')^2\sum_p \frac{|\varphi_p|^2}{1+\varphi_{p\bar{p}}} - 2\gamma',
\end{aligned}$$

where we assume that $\gamma' > 0$. Also from Lemma 2.1 we know that $\dot{\varphi}$ is bounded. Combining these estimates we obtain

$$0 \leqslant \gamma''\sum_p \frac{|\varphi_p|^2}{1+\varphi_{p\bar{p}}} - (\gamma' - Bt)\sum_p \frac{1}{1+\varphi_{p\bar{p}}} + \log\beta + \frac{Ct}{\beta} + C\gamma' + Ct.$$

We now choose $\gamma(s) = As - \frac{1}{A}s^2$. We can assume that $\log\beta > 1$ at (t, z), so in particular $\frac{t}{\beta}$ is bounded above as long as $t < T$. Then if A is chosen sufficiently large, we get a constant $C' > 0$ such that

$$\sum_p \frac{1}{1+\varphi_{p\bar{p}}} + \sum_p \frac{|\varphi_p|^2}{1+\varphi_{p\bar{p}}} \leqslant C'\log\beta, \tag{8}$$

so in particular $(1+\varphi_{p\bar{p}})^{-1} \leqslant C'\log\beta$ for each p. From (4) we know that

$$\prod_p (1+\varphi_{p\bar{p}}) < C,$$

so

$$1+\varphi_{p\bar{p}} \leqslant C(C'\log\beta)^{n-1},$$

and using (8) we get

$$\beta = \sum_p |\varphi_p|^2 \leqslant C(C'\log\beta)^n.$$

This shows that $\beta < C$ and in turn $K < C$ for some constant C. So either K achieves a maximum for some $t > 0$ in which case we have just bounded it, or it achieves its maximum for $t = 0$, which is bounded in terms of $\sup|\varphi_0|$. $\square$

From now on, let us write g for the metric ω and g_φ for the metric $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$.

Lemma 2.3. *There exists $C > 0$ depending on* $\sup|\varphi_0|$ *and* $\sup|\dot{\varphi}_0|$ *such that*

$$0 < \mathrm{tr}_g(g_\varphi) = n + \Delta_g \varphi(t) < e^{Ce^{C/t}}, \tag{9}$$

as long as the solution exists and $t \leqslant T$, for the T from Lemma 2.1.

Proof. We let

$$H = e^{-\frac{\alpha}{t}} \log \mathrm{tr}_g(g_\varphi) - A\varphi,$$

where $\alpha = C$ from Lemma 2.2 and A is chosen later. In particular we will use that $e^{-\alpha/t}|\nabla\varphi|_g^2 < 1$. Standard calculations (from Aubin and Yau [1, 19]) show that there exist $B > 0$ such that

$$\Delta_\varphi \log \mathrm{tr}_g(g_\varphi) \geqslant -B\mathrm{tr}_{g_\varphi} g - \frac{\mathrm{tr}_g \mathrm{Ric}(g_\varphi)}{\mathrm{tr}_g(g_\varphi)}.$$

Using this we can compute

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_\varphi\right) H \leqslant& \frac{\alpha e^{-\alpha/t}}{t^2} \log \mathrm{tr}_g(g_\varphi) + \frac{Ce^{-\alpha/t}}{\mathrm{tr}_g(g_\varphi)} + \frac{e^{-\alpha/t}\Delta_g F(\varphi, z)}{\mathrm{tr}_g(g_\varphi)} \\ &+ Be^{-\alpha/t}\mathrm{tr}_{g_\varphi} g - A\dot{\varphi} + An - A\mathrm{tr}_{g_\varphi} g.\end{aligned} \tag{10}$$

Here

$$\Delta_g F(\varphi, z) = \Delta_g F + 2\mathrm{Re}(g^{i\bar{j}} F_i' \varphi_{\bar{j}}) + F'\Delta_g \varphi + F''|\nabla\varphi|_g^2,$$

where F' is the derivative in the φ variable, and $\Delta_g F$ is the Laplacian of $F(\varphi, z)$ in the z variable. So we have constants C_1, C_2, C_3 such that

$$\Delta_g F(\varphi, z) \leqslant C_1 + C_2|\nabla\varphi|_g^2 + C_3\mathrm{tr}_g(g_\varphi).$$

From (4) we have bounds on above and below on $\frac{\det g_\varphi}{\det g}$, so for some constant C we have $\mathrm{tr}_g(g_\varphi) > C^{-1}$ and also $\mathrm{tr}_g(g_\varphi) \leqslant C(\mathrm{tr}_{g_\varphi} g)^{n-1}$. Using these in (10) we get

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_\varphi\right) H &\leqslant -(A - Be^{-\alpha/t})\mathrm{tr}_{g_\varphi} g + C \log \mathrm{tr}_{g_\varphi} g + C \\ &\leqslant -(A - C - Be^{-\alpha/t})\mathrm{tr}_{g_\varphi} g + C',\end{aligned}$$

as long as $t \leqslant T$. Choosing A large enough, we can use the maximum principle to bound H in terms of its value for $t = 0$, which is bounded by $\sup|\varphi_0|$. □

We note here that if one is interested in the special case of weak Kähler-Einstein currents (i.e. $F = \varphi - h_\omega$), then the gradient estimate in Lemma 2.2 is not needed. We now describe how to get the higher order estimates, as long as the solution exists and $t \leqslant T$, for the T from Lemma 2.1. As in [19] we let $\varphi_{i\bar{j}k}$ be the third covariant derivative of φ with respect to the Levi-Civita connection of ω, and we define

$$S = g_\varphi^{i\overline{p}} g_\varphi^{q\bar{j}} g_\varphi^{k\overline{r}} \varphi_{i\bar{j}k} \varphi_{\overline{p}q\overline{r}}.$$

From now on, we will denote by $C(t)$ a smooth real function defined on $(0, T]$, which is allowed to blow up when t approaches zero, which depends only on $\sup|\varphi_0|, \sup|\dot{\varphi}_0|$ and which may vary from line to line. These functions $C(t)$ can be made completely explicit. Using (9) it is clear that an estimate of the form $S \leqslant C(t)$ implies an estimate of the form $\|\varphi(t)\|_{C^{2+\alpha}(g)} \leqslant C(t)$, for any $0 < \alpha < 1$. To estimate S we first compute its evolution. It is convenient to use the general computation by Phong-Šešum-Sturm [14], which uses the following notation. We denote by $h_j^i =$

$g^{i\bar{k}}(g_{j\bar{k}}+\varphi_{j\bar{k}})$, which is an endomorphism of the tangent bundle. Then S can be written in terms of the connection ∇hh^{-1} as

$$S = g_{\varphi}^{p\bar{q}} g_{\varphi,i\bar{j}} g_{\varphi}^{k\bar{\ell}} (\nabla_p hh^{-1})^i_k \overline{(\nabla_q hh^{-1})^j_\ell} = |\nabla hh^{-1}|^2_{g_\varphi},$$

where ∇ is the Levi-Civita connection of ω_φ. Then the computations in [14] yield

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_\varphi\right) S = & -|\nabla(\nabla hh^{-1})|^2_{g_\varphi} - |\overline{\nabla}(\nabla hh^{-1})|^2_{g_\varphi} \\ & + 2\mathrm{Re}\left\langle (\nabla T - \nabla R, \nabla hh^{-1}\right\rangle_{g_\varphi} \\ & + (\nabla_p hh^{-1})^i_k \overline{(\nabla_q hh^{-1})^j_\ell} (T^{p\bar{q}} g_{\varphi,i\bar{j}} g_\varphi^{k\bar{\ell}} - g_\varphi^{p\bar{q}} T_{i\bar{j}} g_\varphi^{k\bar{\ell}} + g_\varphi^{p\bar{q}} g_{\varphi,i\bar{j}} T^{k\bar{\ell}}),\end{aligned}$$

where $T_{i\bar{j}} = -\left(\frac{\partial}{\partial t} g_\varphi + \mathrm{Ric}(g_\varphi)\right)_{i\bar{j}}$, $(\nabla T)^p_{qr} = g_\varphi^{p\bar{s}} \nabla_q T_{r\bar{s}}$, $(\nabla R)^p_{qr} = g_\varphi^{s\bar{t}} \nabla_s R^p_{rq\bar{t}}$ and $R^p_{rq\bar{t}}$ is the curvature of the fixed metric g. Along the standard Kähler-Ricci flow the tensor T vanishes, while in our case differentiating (1) we get

$$-T_{i\bar{j}} = \mathrm{Ric}(g)_{i\bar{j}} + F''\varphi_i\varphi_{\bar{j}} + F'\varphi_{i\bar{j}} + F_{i\bar{j}} + 2\mathrm{Re}(F'_i\varphi_{\bar{j}}). \tag{11}$$

Using (7) and (9) we can then estimate

$$|(\nabla_p hh^{-1})^i_k \overline{(\nabla_q hh^{-1})^j_\ell} (T^{p\bar{q}} g_{\varphi,i\bar{j}} g_\varphi^{k\bar{\ell}} - g_\varphi^{p\bar{q}} T_{i\bar{j}} g_\varphi^{k\bar{\ell}} + g_\varphi^{p\bar{q}} g_{\varphi,i\bar{j}} T^{k\bar{\ell}})| \leqslant C(t) S.$$

The term $2\mathrm{Re}\langle \nabla R, \nabla hh^{-1}\rangle_{g_\varphi}$ is comparable to S, but bounding $2\mathrm{Re}\langle \nabla T, \nabla hh^{-1}\rangle_{g_\varphi}$ requires a bit more work. Differentiating (11) and using (3), (7) and (9) we see that all the terms in $2\mathrm{Re}\langle \nabla T, \nabla hh^{-1}\rangle_{g_\varphi}$ are comparable to $C(t)S$ except for two terms of the form

$$\langle \varphi_{ij} g_\varphi^{k\bar{\ell}} \varphi_{\bar{\ell}}, (\nabla_i hh^{-1})^k_j \rangle_{g_\varphi}.$$

We bound these by $|\varphi_{ij}|^2_{g_\varphi} + C(t)S$, so overall we get

$$\left(\frac{\partial}{\partial t}-\Delta_\varphi\right) S \leqslant C(t)S + |\varphi_{ij}|^2_{g_\varphi} + C.$$

The term $C(t)S$ can be controlled by using $\mathrm{tr}_g(g_\varphi)$ in the usual way (cfr. [14]). For the term $|\varphi_{ij}|^2_{g_\varphi}$ we note that using (3), (7) and (9) we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_\varphi\right) |\nabla\varphi|^2_g & \leqslant -\sum_{i,p} \frac{|\varphi_{ip}|^2 + |\varphi_{i\bar{p}}|^2}{1+\varphi_{p\bar{p}}} + 2\mathrm{Re}\langle \nabla\varphi, F'\nabla\varphi + \nabla F\rangle_g + C\mathrm{tr}_{g_\varphi} g |\nabla\varphi|^2_g \\ & \leqslant -\frac{|\varphi_{ij}|^2_{g_\varphi}}{C(t)} + C(t).\end{aligned}$$

We can then apply the maximum principle to the quantity

$$G = \frac{S}{C_1(t)} + \frac{\mathrm{tr}_g(g_\varphi)}{C_2(t)} + \frac{|\nabla\varphi|^2_g}{C_3(t)},$$

for suitable functions $C_i(t)$ that depend only on the given data, and get $G \leqslant C$, which implies the desired estimate for S. This means that as long as the solution exists and $0 < t \leqslant T$ we have a bound on $\|\varphi(t)\|_{C^{2+\alpha}(M)}$. Since by standard parabolic theory one can start the flow with initial data in $C^{2+\alpha}$, this shows that the flow has a $C^{2+\alpha}$ solution defined on $[0, T]$.

The next step is to estimate $\sup|\ddot{\varphi}(t)|$ and $\sup|\partial_i\partial_{\bar{j}}\dot{\varphi}(t)|$. It is easy to see that both of these quantities are bounded if we bound $|\mathrm{Ric}(g_\varphi)|_{g_\varphi}$. Following the computation in [15, (6.31)] one can derive the following estimate (there are essentially no new bad terms in this case)

$$\left(\frac{\partial}{\partial t}-\Delta_\varphi\right)|\mathrm{Ric}(g_\varphi)|_{g_\varphi}\leqslant C(t)|\mathrm{Rm}(g_\varphi)|^2+C(t).$$

From one of the two good positive terms in the evolution of S we get

$$\left(\frac{\partial}{\partial t}-\Delta_\varphi\right)S\leqslant-\frac{|\mathrm{Rm}(g_\varphi)|^2}{C(t)}+C(t)$$

and so the maximum principle applied to the quantity $\frac{|\mathrm{Ric}(g_\varphi)|_{g_\varphi}}{C_1(t)}+\frac{S}{C_2(t)}$ gives the desired bound $|\mathrm{Ric}(g_\varphi)|_{g_\varphi}\leqslant C(t)$.

It now follows from the parabolic Schauder estimates applied to (5) that we have bounds for φ in the parabolic Hölder space $C^{2+\alpha,1+\alpha/2}(M\times[\varepsilon,T])$ for any $\varepsilon>0$, with the bounds only depending on ε, $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$. By the parabolic Schauder estimates we then also get bounds on all higher order derivatives for φ, and letting $\varepsilon\to 0$ we get the required bounds on $\varphi(t)$ that blow up as t goes to zero. In particular, we get a smooth solution $\varphi(t)$ that exists on $[0,T]$, with bounds as in (2). This completes the proof of Proposition 2.1.

3. Proof of Theorem 1.1

Suppose that φ is a bounded ω-plurisubharmonic solution of the equation

$$(\omega+\sqrt{-1}\partial\overline{\partial}\varphi)^n=e^{-F(\varphi,z)}\omega^n,\tag{12}$$

where F is a smooth function. First of all we want to prove existence of the flow (1) with rough initial data φ. For this, we follow the proof of Song-Tian [18] in the case of Kähler-Ricci flow.

It follows from Kołodziej [11] that in this case φ is continuous (in fact it is even C^α [9, 13]). Let us approximate φ with a sequence of smooth functions u_k, such that

$$\sup_M|\varphi-u_k|\to 0,\tag{13}$$

as $k\to\infty$. By Yau's theorem [19] there are smooth functions ψ_k such that

$$(\omega+\sqrt{-1}\partial\overline{\partial}\psi_k)^n=c_ke^{-F(u_k,z)}\omega^n,\tag{14}$$

where the positive constants c_k are chosen so that the integrals of both sides of (14) match. When k is large we see that c_k approaches 1. Moreover, we can normalise the solution ψ_k so that

$$\sup_M(\psi_k-\varphi)=\sup_M(\varphi-\psi_k).$$

Using (13) together with Kołodziej's stability result [12] we obtain

$$\lim_{k\to\infty}\|\psi_k-\varphi\|_{L^\infty}=0.\tag{15}$$

Using Proposition 2.1 we can solve the equation

$$\frac{\partial\varphi_k}{\partial t}=\log\frac{(\omega+\sqrt{-1}\partial\overline{\partial}\varphi_k)^n}{\omega^n}+F(\varphi_k,z)-\log c_k,\tag{16}$$

with initial condition $\varphi_k|_{t=0} = \psi_k$ for a short time $t \in [0,T]$ independent of k, since by (13), (14) and (15) we have uniform bounds on the initial data $\sup|\psi_k|$ and $\sup|\dot{\varphi}_k(0)|$. As in [18] we have

Lemma 3.1. *The sequence φ_k is a Cauchy sequence in $C^0([0,T]\times M)$, ie.*

$$\lim_{j,k\to\infty} \|\varphi_j - \varphi_k\|_{L^\infty([0,T]\times M)} = 0.$$

Proof. Fix j,k and let $\mu = \varphi_j - \varphi_k$. Then

$$\frac{\partial\mu}{\partial t} = \log\frac{(\omega+\sqrt{-1}\partial\overline{\partial}\varphi_k + \sqrt{-1}\partial\overline{\partial}\mu)^n}{(\omega+\sqrt{-1}\partial\overline{\partial}\varphi_k)^n} + F(\varphi_j,z) - F(\varphi_k,z) + \log\frac{c_k}{c_j},$$

and $\mu|_{t=0} = \psi_j - \psi_k$. At any time given time t, the maximum of μ is achieved at some point $z\in M$, and at z we have

$$\frac{d\mu_{\max}}{dt} \leqslant F(\varphi_j(t,z),z) - F(\varphi_k(t,z),z) + \log\frac{c_k}{c_j} \leqslant \kappa|\mu(z)| + \log\frac{c_k}{c_j},$$

where κ is independent of j,k. Similarly, at the point z' where the minimum of μ is achieved, we have

$$\frac{d\mu_{\min}}{dt} \geqslant -\kappa|\mu(z')| + \log\frac{c_k}{c_j}.$$

Putting these together we see that

$$\frac{d|\mu|_{\max}}{dt} \leqslant \kappa|\mu|_{\max} + \left|\log\frac{c_k}{c_j}\right|,$$

where the derivative is interpreted as the limsup of the difference quotients at the points where it does not exist. It follows that

$$\sup_{[0,T]\times M}|\varphi_j - \varphi_k| \leqslant e^{\kappa T}\left(\|\psi_j-\psi_k\|_{L^\infty(M)} + \frac{1}{\kappa}\left|\log\frac{c_k}{c_j}\right|\right) - \frac{1}{\kappa}\left|\log\frac{c_k}{c_j}\right|.$$

Now (15) and the fact that c_k converges to 1 imply the result. ☐

Using this lemma we can define

$$\Phi = \lim_{j\to\infty}\varphi_j,$$

which is in $C^0([0,T]\times M)$. Moreover from Proposition 2.1 for any $\varepsilon > 0$ we have uniform bounds on all derivatives of the φ_j for $t\in[\varepsilon,T]$, so in fact for all k we have

$$\lim_{j\to\infty}\|\Phi - \varphi_j\|_{C^k(M\times[\varepsilon,T])} = 0.$$

From Equation (6) we get

$$\sup_M|\dot{\varphi}_k(t)| < C\sup_M|\dot{\varphi}_k(0)|$$

for $t\in[0,T)$, but from (16) we have

$$\dot{\varphi}_k(0) = \log\frac{(\omega+\sqrt{-1}\partial\overline{\partial}\psi_k)^n}{\omega^n} + F(\psi_k,z) - \log c_k = F(\psi_k,z) - F(\varphi_k,z) - \log c_k,$$

which converges to zero when k goes to infinity. It follows that for any $t > 0$ we have

$$\dot{\Phi}(t) = \lim_{j\to\infty}\dot{\varphi}_j(t) = 0.$$

Hence Φ is constant on $(0, T]$, but since it is continuous on $[0, T]$ it follows that $\Phi(t) = \Phi(0)$ for all $t \leqslant T$. But $\Phi(0)$ is our solution φ of Equation (12), whereas $\Phi(t)$ is smooth for $t > 0$. Hence φ is smooth.

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